

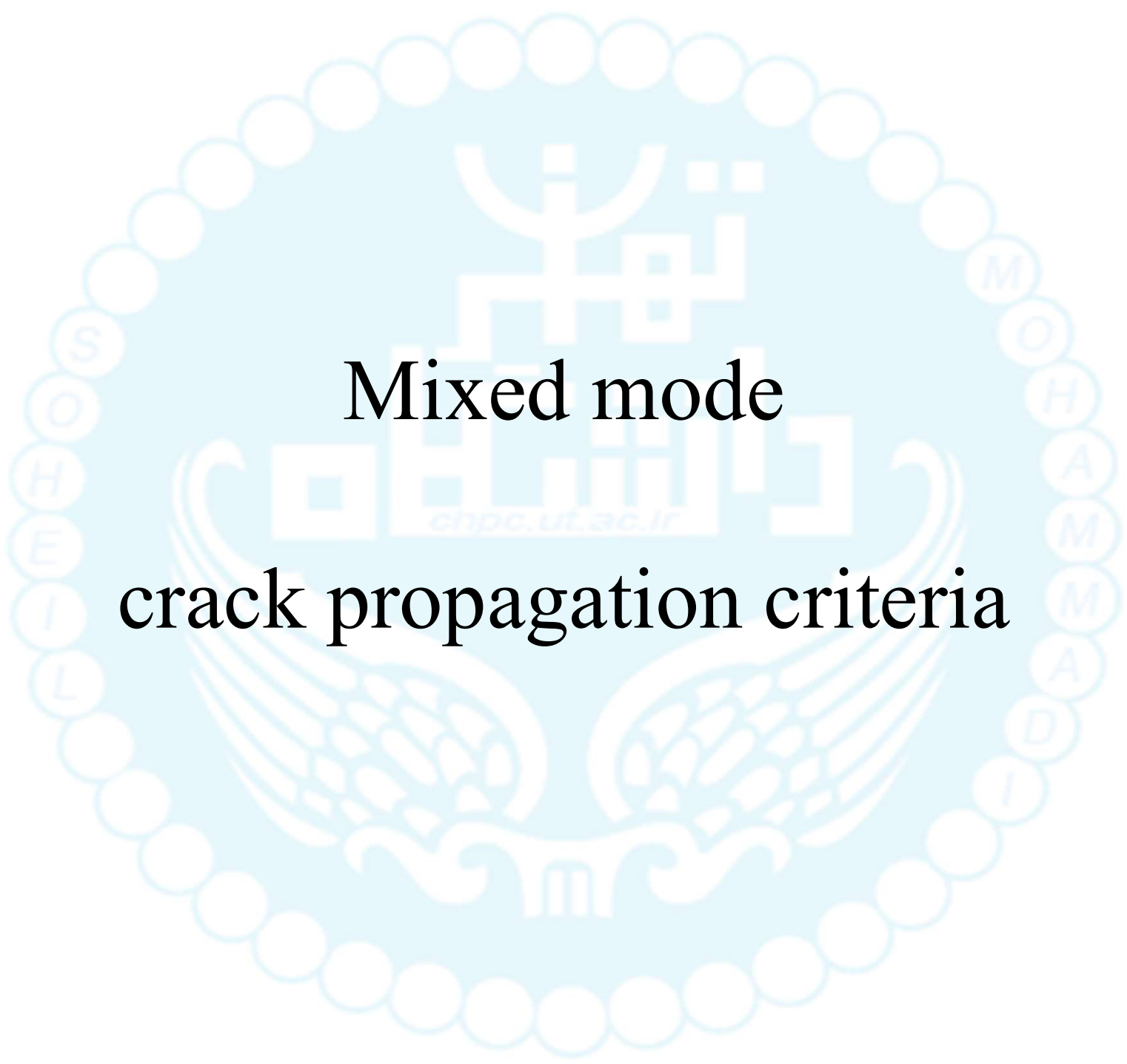


Fracture Mechanics

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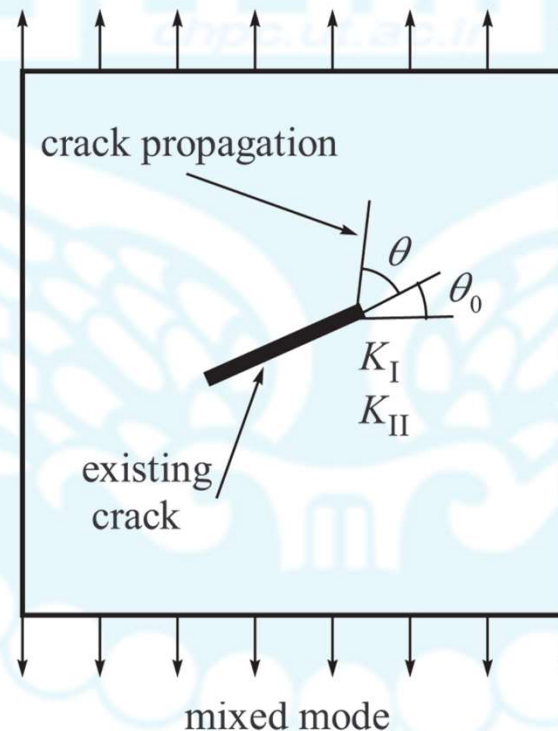
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Mixed mode crack propagation criteria

Mixed mode problems

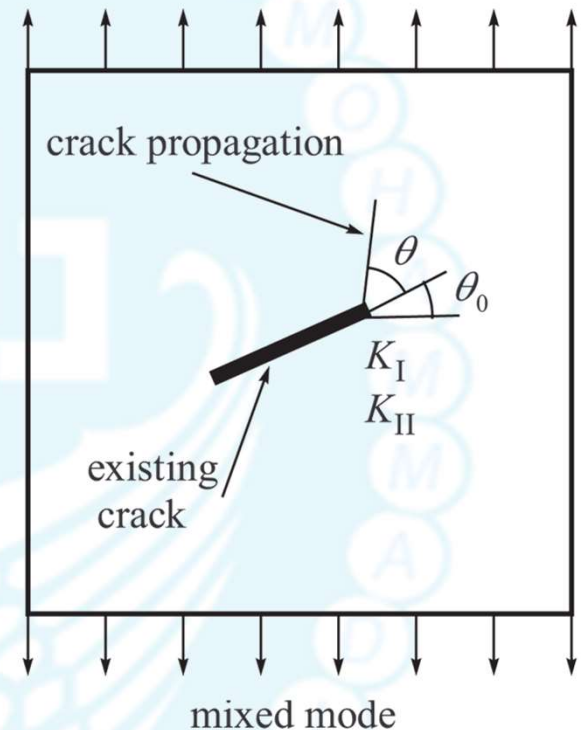
The problems discussed so far were mainly in fracture mode I. Practical engineering problems, however, rarely fall into this category. They usually include inclined or curvilinear crack propagations and are subjected to multiaxial loadings, creating non-zero K_I and K_{II} stress intensity factors. The effect of the tearing mode is neglected in this section.



In an infinite uniaxial tensile plate with an inclined crack, the mode I and II stress intensity factors can be written as,

$$K_I = \sigma_0 \cos^2 \theta_0 \sqrt{\pi a}$$

$$K_{II} = \sigma_0 \sin \theta_0 \cos \theta_0 \sqrt{\pi a}$$



Clearly, these simple rules cannot be used in general mixed mode problems.

For a crack to extend in a linear elastic flawed structure, two types of criteria can be considered:

1. First, a global approach for comparing the energy release rate, G , determined from the crack global transfer of energy, with a material property, G_c (critical energy release rate).
2. The second approach is based on a rather local criterion by comparing the stress intensity factor, K , determined from the near crack tip stress field, with a material property K_c (critical stress intensity factor or fracture toughness).

Generalisation of the original SIF-based collinear crack propagation criterion ($K_I > K_{Ic}$) to include mixed mode effects can be defined in terms of K_I and K_{II} stress intensity factors, and K_{Ic} and K_{IIc} fracture toughness factors:

$$f(K_I, K_{Ic}, K_{II}, K_{IIc}) = 0$$

This equation is further simplified because usually only the mode I fracture toughness K_{Ic} is experimentally measured:

$$f(K_I, K_{Ic}, K_{II}) = 0$$

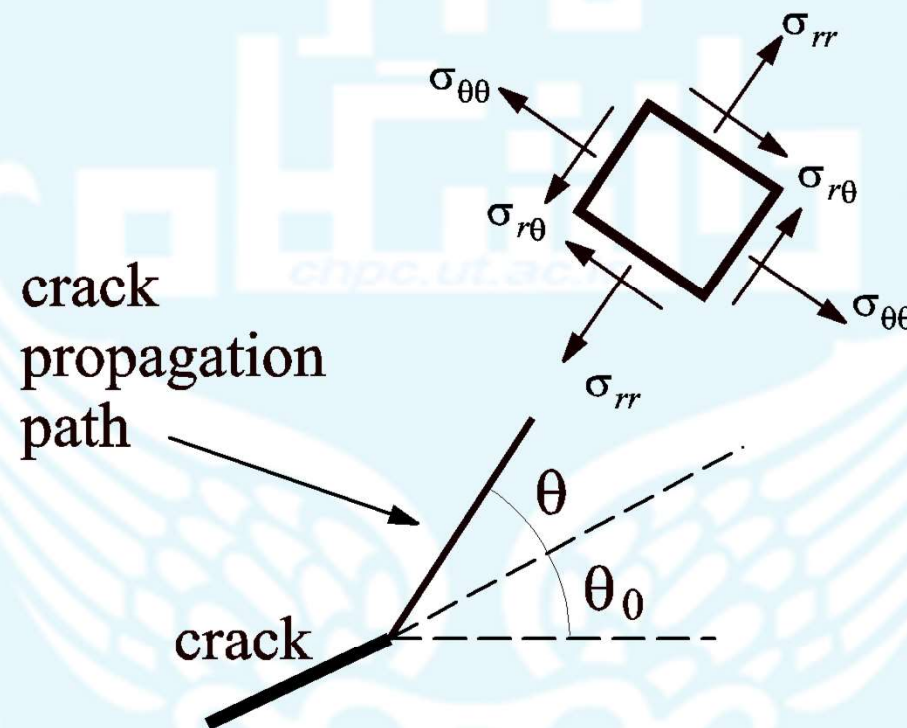
In the following, some of the most widely used mixed mode criteria are briefly reviewed.

Maximum circumferential/hoop tensile stress

Erdogan and Sih (1963) developed the first theory of mixed mode stress intensity factor based on the solution of stress state near a crack tip.

While the method is originally developed for ideal isotropic homogeneous static crack propagation in simple conditions, it is frequently used for many rather complicated problems, at least for first evaluations. This is, in fact, the most frequently used mixed mode fracture criterion.

They assumed that the crack propagates at its tip in a radial direction within a plane perpendicular to the direction of maximum tension when the maximum circumferential tensile stress $(\sigma_{\theta\theta})_{\max}$ reaches a critical material constant.



According to this criterion, the crack propagates in mode I along the new propagation path.

The mixed mode criterion for a crack angle θ can then be defined as

$$K_{\theta\theta} = K_{Ic}$$

$$K_{\theta\theta} = \sigma_{\theta\theta} \sqrt{2\pi r} = K_I \cos^3 \frac{\theta}{2} - \frac{3}{2} K_{II} \cos \frac{\theta}{2} \sin \theta$$

As mentioned earlier, this method is usually used for any new type of fracture problems, where no dedicated mixed mode fracture criterion is yet to be developed.

Note that $K_{\theta\theta}$ may become smaller or greater than K_I , as K_{II} may be positive or negative, while K_I remains always positive. Also, note that θ may vary from $[0-2\pi]$.

Minimum strain energy density criterion

The minimum strain energy density criterion is based on this idea that a crack propagates along the minimum resistance path. It determines the crack propagation from the crack tip in a direction θ , along which the strain energy density at a critical distance is minimum.

Occurrence of crack propagation is controlled by checking such a minimum resistance until it reaches a critical value.

This criterion, in concept, may well be extended to nonlinear fracture mechanics problems. Here, however, it is only explicitly derived for LEFM problems.

The final form of the criterion according to Sih (1973) can be defined as (Chang, 2006):

$$W_s = \frac{1}{16\pi\mu} \left[a_{11}K_I^2 + 2a_{12}(K_I K_{II}) + a_{22}K_{II}^2 \right]$$
$$\begin{cases} a_{11} = (1 + \cos \theta)(\kappa - \cos \theta) \\ a_{12} = \sin \theta [2 \cos \theta - (\kappa - 1)] \\ a_{22} = (\kappa + 1)(1 - \cos \theta) + (1 + \cos \theta)(3 \cos \theta - 1) \end{cases}$$

and the propagation occurs subject to

$$\text{Min}[W_s(\theta) = W_c] \rightarrow \text{propagation}$$

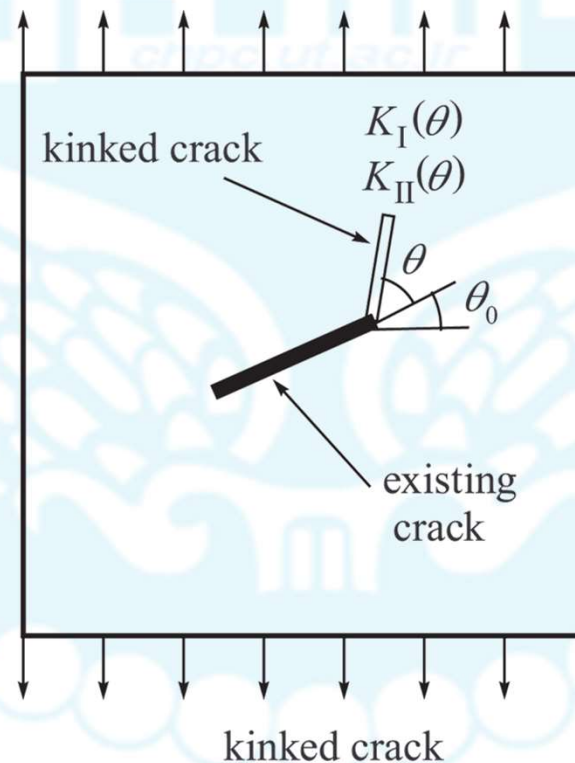
$$\frac{\partial W_s(\theta)}{\partial \theta} = 0 \rightarrow \text{propagation angle}$$

Furthermore, the instability of crack propagation (and not the crack instability) is determined from:

$$\frac{\partial^2 W_s(\theta)}{\partial \theta^2} > 0 \quad \text{instability condition}$$

Maximum energy release rate

An alternative model, the maximum energy release rate, is based on the original work of Erdogan and Sih (1963) and Hussain *et al.* (1974) who solved for the stress intensity factors $K_I(\theta)$ and $K_{II}(\theta)$ of a major existing crack with an infinitesimal tip kink at an angle θ in terms of the stress intensity factors of the original crack K_I and K_{II} .



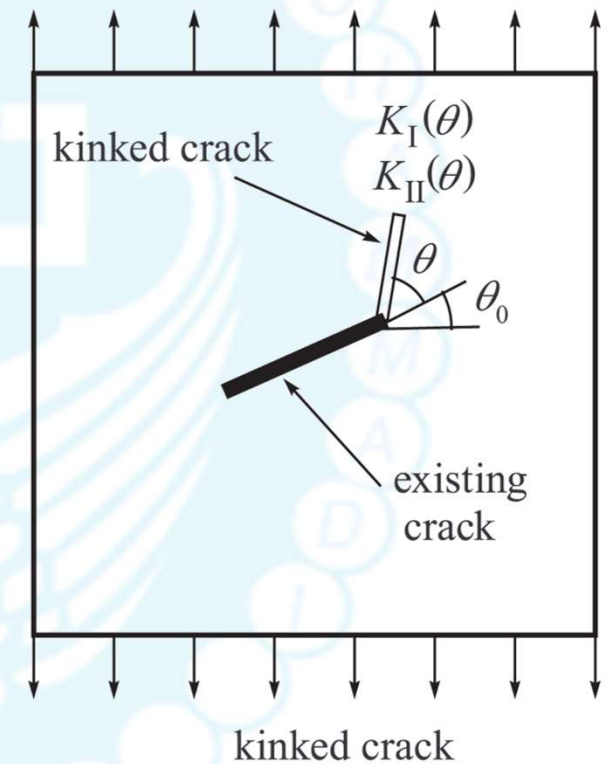
This criterion, in concept, may well be extended to nonlinear fracture mechanics problems. Here, however, it is only explicitly derived for LEFM problems.

For an infinitesimal tip kink at an angle θ

$$K_I(\theta) = g(\theta) \left(K_I \cos \theta + \frac{3}{2} K_{II} \sin \theta \right)$$

$$K_{II}(\theta) = g(\theta) \left(K_{II} \cos \theta - \frac{1}{2} K_I \sin \theta \right)$$

$$g(\theta) = \left(\frac{4}{3 + \cos^2 \theta} \right) \left(\frac{1 - \theta / \pi}{1 + \theta / \pi} \right)^{\frac{\theta}{2\pi}}$$



Adopting Irwin's generalised expression for the energy release rate, evaluation of $G(\theta)$ for the kinked crack becomes

$$\begin{aligned} G(\theta) &= \frac{1}{E'} (K_I^2(\theta) + K_{II}^2(\theta)) \\ &= \frac{1}{4E'} g^2(\theta) \left[(1 + 3 \cos^2 \theta) K_I^2 + 8 \sin \theta \cos \theta K_I K_{II} + (9 - 5 \cos^2 \theta) K_{II}^2 \right] \end{aligned}$$

and the propagation occurs subject to

$$\text{Max}[G(\theta) = G_c] \rightarrow \text{propagation}$$

$$\frac{\partial G(\theta)}{\partial \theta} = 0 \rightarrow \text{propagation angle}$$

Furthermore, the instability of crack propagation (and not the crack instability) is determined from:

$$\frac{\partial^2 G(\theta)}{\partial \theta^2} < 0 \quad \text{instability condition}$$

This method can somehow be interpreted as a higher order approach (similar to the higher order solution of elastic buckling instability, where the solution is sought on the unknown deformed state.

Non-local stress fracture criterion

A **non-local** stress fracture criterion based on the **damage model** of an elastic solid, containing growing micro-cracks, was proposed by Romanowicz and Seweryn (2008).

According to this criterion, crack initiation and propagation would occur when the mean value of the function $R(\sigma_n, \tau_n)$ of decohesive normal and shear stress over a segment d , the length of the damaged zone, reaches its critical value

$$\max_{\theta} \bar{R}(\sigma_n, \tau_n) = \max_{\theta} \left[\frac{1}{d} \int_0^d R(\sigma_n, \tau_n) dx \right] = 1$$

$R(\sigma_n, \tau_n)$ is the local stress function obtained by the micro-crack damage model, which states that the propagation of micro-cracks takes place when the stress energy release rate equals its resistance to micro-crack growth (Hallai, 2008).

Empirical models

The following typical relation defines a number of mixed mode stress intensity criteria that are basically developed from experimental observations rather than theoretical bases.

Many of them are valid only for specific problems in concrete, rock and composites, where the models have been experimentally obtained or calibrated.

They are usually of the general form of:

$$\alpha_1 \left(\frac{K_I}{K_{Ic}} \right)^{\beta_1} + \alpha_2 \left(\frac{K_I K_{II}}{K_{Ic} K_{IIc}} \right)^{\beta_2} + \alpha_3 \left(\frac{K_{II}}{K_{IIc}} \right)^{\beta_3} + \alpha_4 \left(\frac{K_{II}}{K_{Ic}} \right)^{\beta_4} = 1$$

Table Constant parameters of empirical mixed mode criteria.

Model	α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4
Linear	1	-	1	-	1	-	1	-
Elliptical	1	-	1	-	2	-	2	-
Quadratic	1	$c \neq 2$	1	-	2	1	2	-
Jernkvist	1	-	1	-	1	-	2-3.4	-
Awaji/Sato	1	-	1	-	1.6	-	1.6	-
Advani/Lee	1	-	-	0.25	1	-	-	2
Palanisawamy Knauss	1	-	-	1.5	1	-	-	2

Linear

$$\frac{K_I}{K_{Ic}} + \frac{K_{II}}{K_{IIc}} = 1$$

very simple and for the first guess

Elliptical

$$\left(\frac{K_I}{K_{Ic}} \right)^2 + \left(\frac{K_{II}}{K_{IIc}} \right)^2 = 1$$

Frequently used for delamination in composites
Also for concrete

Quadratic

$$\left(\frac{K_I}{K_{Ic}} \right)^2 + c \left(\frac{K_I K_{II}}{K_{Ic} K_{IIc}} \right) + \left(\frac{K_{II}}{K_{IIc}} \right)^2 = 1, c \neq 2$$

used for composites and concrete

Jernkvist

$$\left(\frac{K_I}{K_{Ic}}\right)^2 + \left(\frac{K_{II}}{K_{IIc}}\right)^b = 1, \quad b = 2 - 3.4$$

Frequently used for wood and orthotropic materials

Awaji/Sato

$$\left(\frac{K_I}{K_{Ic}}\right)^{1.6} + \left(\frac{K_{II}}{K_{IIc}}\right)^{1.6} = 1$$

Advani/Lee

$$\left(\frac{K_I}{K_{Ic}}\right) + 0.25 \left(\frac{K_{II}}{K_{Ic}}\right)^2 = 1$$

Palanisawamy
Knauss

$$\left(\frac{K_I}{K_{Ic}}\right) + 1.5 \left(\frac{K_{II}}{K_{Ic}}\right)^2 = 1$$

It is important to note that the empirical criteria:

- are very simple.
- They only determine the onset of propagation.
- They do not usually provide any information on crack direction. Therefore, other stress based criteria should be used to determine the direction of crack propagation.
- They largely depend to supporting experimental data and should be cautiously extended to other materials and even experimental setups.

